Gardens of Eden and Fixed Points in Sequential Dynamical Systems

Mi Yu

February 16, 2015

In week one, we study the definition of a sequential dynamical system and a phase space. A sequential dynamical system contains four key elements: a finite graph, a finite set of states for the vertices, an update rule, and an update order. The phase space is a directed graph that contains all the possible states in the SDS as vertices, and its edges are connected based on the mapping of SDS. This week, we will study two special states—Garden of Eden and fixed points.
Before we start to go into our problems, let me provide some useful definitions that will help us in our later proofs. For the convenient sake, we will use the nor function we defined in the first week to explain some of our definition. In case the readers forgot about nor function, nor function is a function that inputs every vertex and its neighbors. If vertex and its neighbors are all 0, the vertex changes to 1. Otherwise, the vertex changes to 0. Our sequential dynamical system graph and the phase space are the following:

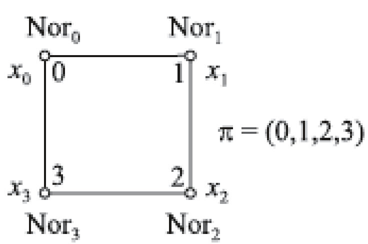

Figure 1: SDS Graph

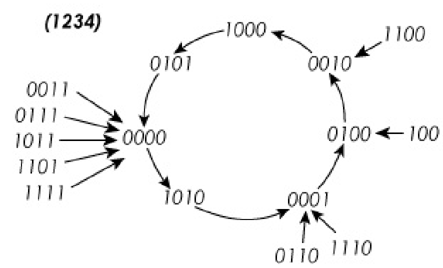

Figure 2: Phase space

Definition 1. *A configuration of an SDS is an n-bit vector $\{b_1, b_2, \dots b_n\}$, where b_i is the value of the state of node v_i $(1 \le i \le n)$.*

So a configuration is what we used to call "possible states" in the first week. The order of the numbers in the vector is determined by the order of the vertices in your SDS graph. Using the nor function example, $\{0, 1, 0, 1\}$ and $\{0, 0, 0, 0\}$ are two configurations.
Now let us denote the function computed by SDS as F_S, which inputs each configuration C to the next configuration C' in the phase space after carrying out the update of node

states based on the update order. For example, in the nor function SDS above, F_S will be a map that sends $\{1,1,0,0\}$ and $\{0,1,0,0\}$ to $\{0,0,1,0\}$, and so on. We can clearly see that this function, comparing to our updating rules, is a tool to study the entire sequential dynamical system instead of just one step update. We can call this kind function a global function, comparing to the update rules, which are local functions. With the knowledge of F_S, we have the following two definitions:

Definition 2. *Given two configuration C' and C of an SDS S, C' is a predecessor of C if* $F_S(C') = C$*, that is, S moves from C' to C in one transition.*

Definition 3. *Given two configurations C' and C of an SDS, C' is an ancestor of C if there is a positive integer t such that* $F_{S^t}(C') = C$*, that is, S evolves from C' to C in one or more transitions.*

There is nothing tricky so far here. If we look at the phase space, the predecessor of any vertex v is the vertex that changes to that vertex v. The ancestor of a vertex v is any vertex in the phase space that eventually can change to the vertex v after few updates. For example, in the nor function SDS, $\{0,0,0,0\}$ is $\{1,0,1,0\}$'s predecessor. $\{0,1,0,1\}$ is one of $\{1,0,1,0\}$'s ancestors. Notice that if a configuration C' is another configuration C's predecessor, then C' is also that configuration C's ancestor. Now you may wonder, as our above example shows, sometimes configurations, such as $\{0,0,1,1\}$ in the above phase space, might not have an ancestor. Of course, if we want to study them, we need to give them a name. Look at the following two definitions:

Definition 4. *A configuration C of an SDS S is a Garden of Eden (GE) configuration if C has no predecessor.*

Definition 5. *A configuration C of an SDS is a fixed point if* $F_S(C) = C$*, that is, if the transition out of C is to C itself.*

Based on the definition, we can say that we find many GE[1] configuration in above example. For example, $\{0,0,1,1\}$ is one of them since no other configurations can change to $\{0,0,1,1\}$. We cannot, however, find a fixed point. Fixed point requires that the configuration has to be the same not only as its ancestor but the same as the predecessor. This means that in our phase space for the SDS, we must have at least one self-loop. Now, let me introduce another example. Looking at the following F_S global map:

0000 ↦ 0000	0001 ↦ 0000
0010 ↦ 0000	0100 ↦ 0000
1000 ↦ 0000	0011 ↦ 0011
0101 ↦ 1111	1001 ↦ 1001
0110 ↦ 0110	1010 ↦ 0000
1100 ↦ 1100	0111 ↦ 1111
1011 ↦ 1111	1101 ↦ 1111
1110 ↦ 1111	1111 ↦ 1111

Figure 3

If we draw the the phase space for this map, we will have:

[1]The abbreviation for Garden of Eden configuration

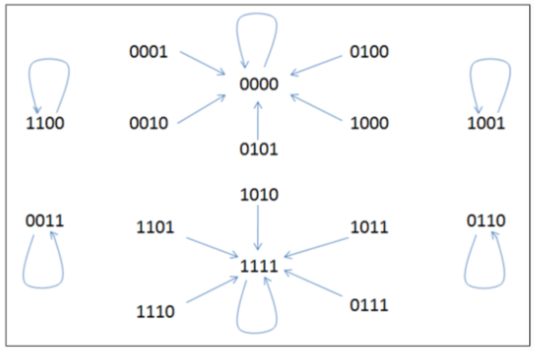

Figure 4

In this phase space, we can find 6 self-loop: $\{1,1,0,0\}$, $\{0,0,1,1\}$, $\{0,0,0,0\}$, $\{1,1,1,1\}$, $\{1,0,0,1\}$, and $\{0,1,1,0\}$. These configurations are fixed points. As I just mentioned that a fixed point must be the same as its predecessor. A good question the readers might ask is that what if a configuration is the same as its ancestor but not predecessor. Please looking at the following definition:

Definition 6. *A configuration C of an SDS is a cycle configuration if C is on a cycle of length 2 or more in the phase space.*

Now based on this definition, we can conclude that if in a phase space, a vertex is sent to itself without using any self-loop, there must exist a cycle configuration somewhere in the phase space. Our figure 3 does not have any cycle configuration. Our original nor function SDS, on the other hand, has a big obit in the middle. All the configurations in the orbit that contain at least two configurations will contain cycle configurations because as we update our SDS, a configuration in an orbit will eventually cycle through the entire orbit and get back to itself[2]. Now since we have covered all the configurations that will eventually change back to itself, what about those will not? Look at the following definition:

Definition 7. *A configuration C of an SDS is a transient configuration if C is neither a fixed point nor a cycle configuration*

Unlike fixed points and cycle configurations, we notice that transient configurations are those vertices in a phase space that will never be revisited again. Notice that GE is a special case of transient configurations. In some sense GE not only will not be revisited again, but GE is also the start point since it does not have any predecessors, That is why it is called the Garden of Eden.
After all these definitions, the readers should have known that the local functions and the global functions play very important roles in determining if there is a fixed points or GE. In order to relate functions to configurations, first of all, let me define the following term:

[2]Based on the definition for an orbit from week 1

Definition 8. *An SDS S is invertible if the function F_S is a bijection.*

Take first look at this definition, and some readers might get confused. Try not to think about the phase space but rather the map F_S. For example, if we slightly modify figure 3, we can make a bijection. We just need to make sure that for every element in the output set, we only have one element in the input set that is mapped to it. As the following modified map shows:

0000 ↦ 0000	0001 ↦ 1110
0010 ↦ 0101	0100 ↦ 0011
1000 ↦ 0010	0011 ↦ 0100
0101 ↦ 1111	1001 ↦ 1001
0110 ↦ 0110	1010 ↦ 1011
1100 ↦ 1100	0111 ↦ 0001
1011 ↦ 1101	1101 ↦ 0100
1110 ↦ 0111	1111 ↦ 1010

Figure 5

Theorem 1. *Let S be a FR-SDS*[3]*. Then the following statements are equivalent:*

1. *S has a transient configuration.*
2. *S has a GE configuration*
3. *S has a configuration with two or more predecessors.*
4. *S is not invertible.*

Proof. First of all, by definition, we know that each configuration only has exactly one successor, or $F_S(C)$. In other word, each configuration can only change to one configuration. Then, we know that if the phase space has N nodes, the number of edges in the phase space is also N. With this knowledge, we can prove our theorem.
First, we will prove that if S has a transient configuration, then S has a GE configuration. There are two cases. The first case is that if C configuration is a transient configuration that has no predecessor, then we are done. The second case is that if C is a transient configuration that has predecessor. Consider the sequence Pred(C), Pred(Pred(C), Pred(Pred(Pred(C))), ... In this sequence, we know that there will be no repeating configuration because if there is a repeating configuration, which means a configuration will either output two different configurations, or the entire phase space is a cycle. However, we know that it is not possible to either output two different configurations or have C to be a cycle configuration. Also, since this is a finite SDS, there will eventually be a configuration that does not have predecessor in the sequence. That configuration is GE configuration. Thus, we proved (1) → (2).
Now let us prove (2) → (3). We can prove this by contradiction. Let us assume that S only has configuration with one or no predecessor. We know that there is a GE configuration.

[3]SDSs with finite phase spaces

GE configuration has 0 predecessor based on the definition. Thus, we have $N-1$ nodes left in our phase space. The maximum directed edges we can have here is $N-1$ edges, assuming that the rest $N-1$ nodes all have one predecessor. This, however, also contradicts to the claim we proved earlier, that there are N directed edges. Thus, based on the method of contradiction, we proved that if S has a GE configuration, S has a configuration with two or more predecessors.
Now let us prove (3) $\rightarrow$ (4). If S has a configuration with two or more predecessors. We know that for at least one configuration, there are two or more configurations that get mapped to it. This means that F_S is not a bijection. Thus, S is not invertible.
At last, we want to prove (4) $\rightarrow$ (1). We know that if S is not invertible, then F_S is not a bijection. In particular, since F_S is a map that maps the inputs to itself. If F_S is not a bijection, then F_S is neither injective nor subjective.[4]. F_S is not surjective is particular useful here. F_S is not surjective implies that there is some configuration that does not have a predecessor. This means that there is a GE configuration, and GE configuration is a special case of transient configurations. □

With (1) $\rightarrow$ (2), (2) $\rightarrow$ (3), (3) $\rightarrow$ (4), and (4) $\rightarrow$ (1), it is sufficient enough to say that the four statements are equivalent.
We currently are working on a SDS version of the game of life. Specifically, we are trying to find a glider or an oscillator in SDS Game of Life. In other words, if we draw the phase space based on the game of life rules, we will try to find a cycle configuration or a fixed point. Meanwhile, we will keep exploring GE configurations and fixed points in SDS.

References

[1] Barrett, Christopher L., Harry B. Hunt, III, Madhav V. Marathe, S. S. Ravi, Daniel J. Rosenkrantz, Richard E. Stearns, and Predrag T. Tosic. *Gardens of Eden and Fixed Points in Sequential Dynamical Systems.* Discrete Mathematics and Theoretical Computer Science Proceedings (2001): 95-110. Web.

[2] Duvall, James M.W. *Characterization of Fixed Points in Sequential Dynamical Systems.* Virginia Polytechnic Institute and State University Department of Mathematics (n.d.): Web.

[3] H. S. Mortveit and C. M. Reidys. *An Introduction to Sequential Dynamical Systems.* Springer Science+Business Media LLC, New York, 2008.

[4] Try to think about why this is true